# The Sum Product Conjecture

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## 1 Abstract

In this paper, we re-derive Fermat's factorization identity through a geometrical approach. We exploit its Pythagorean triangle configuration to investigate the relationship between the sum of two numbers and their product. This visual perspective of numbers enhances our understanding of their nature, equipping us with the tools to tackle unsolved problems in number theory with a completely new approach.

## 2 Introduction

Although addition and multiplication are widely considered to be relatively simple operations in mathematics, little is known about the relationship between the sum of two numbers and their product. This open problem in number theory has been difficult to solve. One breakthrough was the Erdős–Szemerédi Theorem, which was proven in 1983. Rather than directly defining a relationship between the sum and product, Paul Erdős and Endre Szemerédi approached this problem by comparing the sum grid for a set of integers to its product grid.[1]

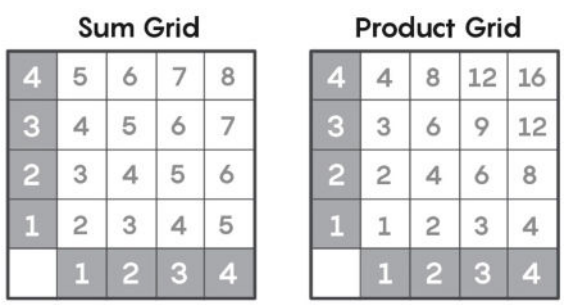


Figure 1: The sum grid and the product grid for the set $A = \{1, 2, 3, 4\}$

The Erdős–Szemerédi Theorem states that for any set of integers $A$,

$$\max(|A + A|, |A \cdot A|) \geq c|A|^{1+\delta}$$

where $c$ is a constant, $\delta$ is a threshhold constant in the range of $0 < \delta < 1$, $|A+A|$ represents the cardinality of the sum grid, and $|A \cdot A|$ represents the cardinality of the product grid, and $|A|$ represents the cardinality of $A$.[2]

Since then, mathematicians have been able to further constrain the value of the $\delta$ threshold. For example, in 1997, György Elekes proved that $|A + A| + |A \cdot A|$ is at least $|A|^{5/4}$, and in 2009, Jozsef Solymosi proved that $|A + A| + |A \cdot A|$ is at least $|A|^{4/3}$.[1] Their method utilizes a geometrical approach, known as *incidence geometry*, which studies how often specific lines that depend on the integers involved would intersect the distinct sums and products on the grid. Incidence geometry has become the standard method used by mathematicians investigating this problem.

Erdős and Szemerédi also made other observations about sum grids and product grids. They found that if the sum grid for a set of integers has many distinct entries, then the set's product grid has few distinct entries. Likewise, if the product grid for a set of integers has many distinct entries, then the set's sum grid has few distinct entries. In addition, they found that for a set of consecutive integers of size $N$, the number of distinct entries in its sum grid is $2N+1$ and the number of distinct entries in its product grid is $cN$ for some constant $c$. They even found distinctions between the sum and product grids of sets of arithmetic progressions and sum and product grids of sets of geometric progressions.[1]

While these findings certainly indicate that there is an inherent relationship between the sum of two numbers and their product, this research is far from formally defining this relationship. In fact, all current research on the sum product conjecture has been limited to analysis of their respective grids.

Geometrically speaking, the operations of sum and product evoke the concepts of computing the area $A$ and perimeter $P$ of a rectangle or a square. For a rectangle or square with length $x$ and width $y$, $P = 2(x + y)$ and $A = xy$. There is an extra factor of 2 for the perimeter, but the main concept is the same. Further insight comes from the relationship between a rectangle and a square with either the same perimeter or area. Consider a square and a rectangle with the same perimeter; the difference in the areas allows us to derive an equation that combines the sum and product of the sides through a right triangle relationship between the geometric and arithmetic means.

The difference in the areas is calculated as follows:

$\Delta A = A_2 - A_1$

$\Delta A = z^2 - xy$

$\Delta A = (\frac{x+y}{2})^2 - xy$

$\Delta A = (\frac{x-y}{2})^2$

This is a Pythagorean relationship that relates the arithmetic mean, $\frac{x+y}{2}$, to the geometric mean, $\sqrt{xy}$, through the factor of the mean difference, $\frac{x-y}{2}$. This is identical to Fermat's factorization identity,

$N = x \cdot y = \frac{x+y}{2} - \frac{x-y}{2}$.

However, involving the geometry of the relationship opens doors to far-reaching implications and applications.

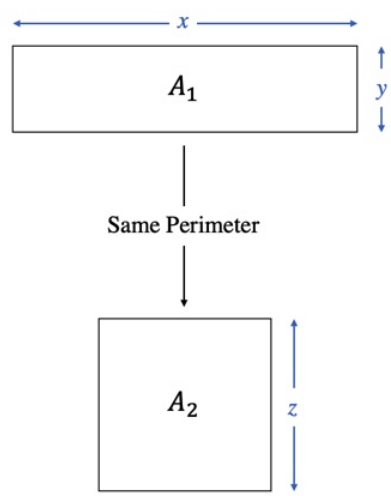


Figure 2: Squaring the rectangle while keeping the perimeter constant.

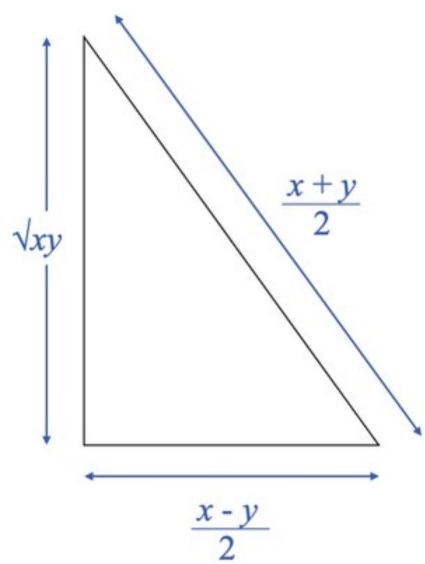


Figure 3: The relationship between the sum and product can be visualized as a right triangle involving the arithmetic mean, geometric mean, and mean difference.

This same triangle of the means emerges naturally from the geometric solution of squaring the rectangle (finding the square of the same area as the rectangle), as shown below:

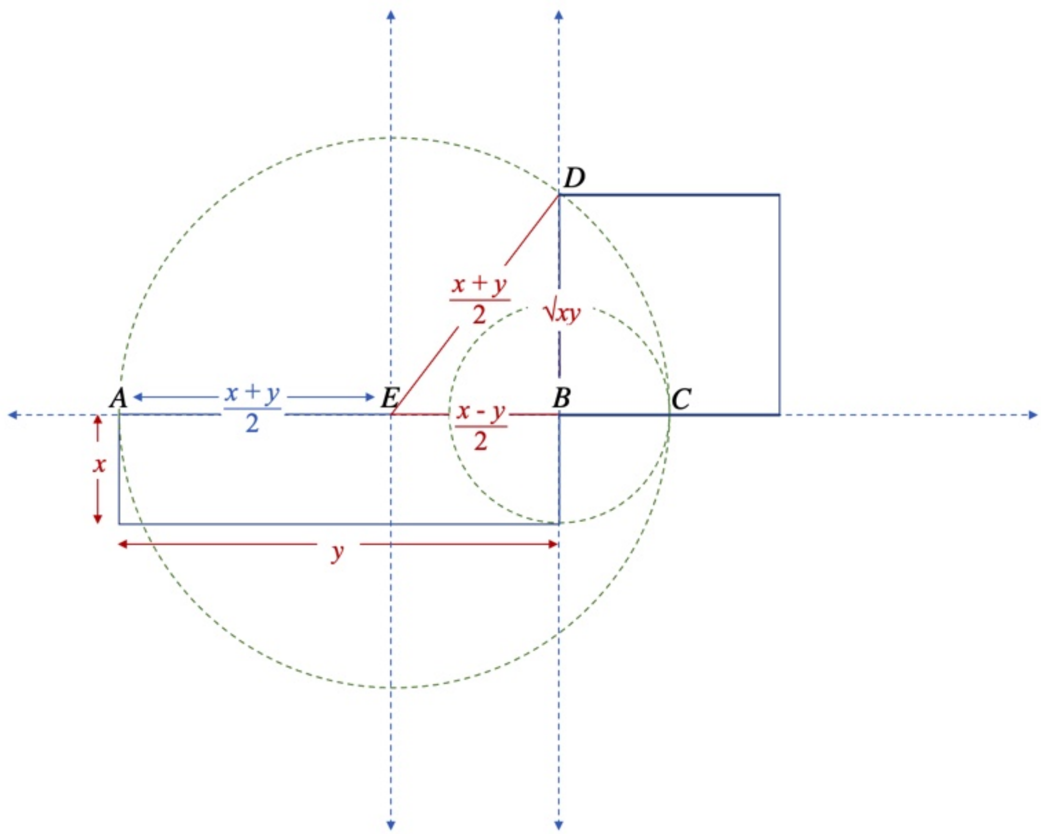


Figure 4: Squaring the rectangle geometrically. We start with a rectangle of sides $x$ and $y$. We extend the $AB$ line horizontally. We then draw a circle of radius $x$, centered at $B$. The circle intersects the horizontal line at $C$. Now, we draw a circle of diameter $AC$ and centered at the midpoint $E$. The point $D$, where the new circle intersects the vertical line $BD$ determines one corner of the square of sides $z$, which can then be easily completed. The triangle $DEB$ is the same triangle of the means.

Thus, the relationship between the sum and product of two numbers can be formally defined, using the Pythagorean Theorem. Every pair of numbers corresponds to a unique right triangle. The sum is used to construct the hypotenuse, the product is used to construct the height, and the difference is used to construct the base.

For any pair of integers $x$ and $y$:

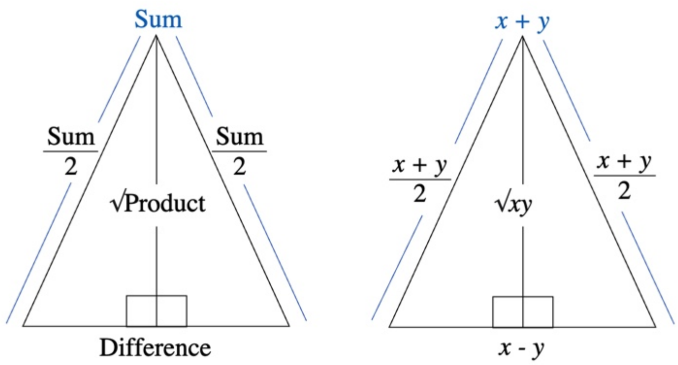


Figure 5: Isosceles triangles made up of two right triangles each, representing the relationship between the sum, difference, and product of two numbers

## 3 Review

Before beginning our proof, it may be helpful to review some important definitions and properties of integers:

<u>**Definition:**</u> **Even Number**
An even number $u$ is an integer such that $u = 2v$ for some integer $v$.

<u>**Definition:**</u> **Odd Number**
An odd number $u'$ is an integer such that $u' = 2v' + 1$ for some integer $v'$.

<u>**Property:**</u> **Integers are closed under addition**
For any pair of integers $w$ and $z$, their sum $w + z$ will always be an integer.

<u>**Property:**</u> **Integers are closed under subtraction**
For any pair of integers $w'$ and $z'$, their difference $w'$ - $z'$ will always be an integer.

<u>**Property:**</u> **Integers are closed under multiplication**
For any pair of integers $w''$ and $z''$, their product $w'' \cdot z''$ will always be an integer.

## 4 Proof

Every integer can be expressed as a product of two factors, which we will refer to as $x$ and $y$. We will prove that every integer can be visualized as at least one right triangle, where each pair of its factors, $x$ and $y$, is used to construct its corresponding right triangle. More specifically, the the sum of $x$ and $y$ will be used to construct the hypotenuse, the product of $x$ and $y$ will be used to construct the height, and the difference of $x$ and $y$ will be used to construct the base.

Consider the Pythagorean Theorem,
$a^2 + b^2 = c^2$
where $a$ and $b$ are both perpendicular side lengths, and $c$ is the hypotenuse of a right triangle.

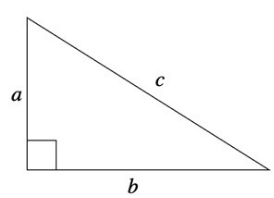


Figure 6: Right triangle with sides $a$, $b$, and $c$

We can rewrite this relationship to solve for $a^2$
$a^2 + b^2 = c^2$
$a^2 = c^2 - b^2$
$a^2 = (c + b)(c - b)$

Let $x = c + b$ and $y = c - b$. Given $c$ and $b$ are both rational numbers such that:

- $2b$ is an integer
- $2c$ is an integer
- $b + c$ is an integer

We will prove that $x$ and $y$ are both integers. Furthermore, since

$a^2 = (c + b)(c - b)$

$a^2 = xy$

This will allow us to conclude that $a^2$ is always an integer as well, since integers are closed under multiplication.

Given $2b$ is an integer, consider the following cases:

**Case 1:** $2b$ is even

If $2b$ is even, then there exists some integer $m$ such that $2b = 2m$. Then,

$2b = 2m$

$b = m$

so $b$ is an integer. Given $b + c$ is some integer $n$,

$b + c = n$

$c = n - b$

We can conclude that $c$ is also an integer, since integers are closed under subtraction. Therefore, $x = c + b$ and $y = c - b$ are both integers, since integers are closed under both addition and subtraction.

**Case 2:** $2b$ is odd

If $2b$ is odd, then there exists some integer $m$ such that $2b = 2m + 1$. Then,

$2b = 2m + 1$

$b = \frac{2m+1}{2}$

$b = m + \frac{1}{2}$

Given $b + c$ is some integer $n$

$b + c = n$

$c = n - b$

$c = n - (m + \frac{1}{2})$

$c = n - m - \frac{1}{2}$

Therefore,

$x = c + b$

$x = (n - m - \frac{1}{2}) + (m + \frac{1}{2})$

$x = n - m - \frac{1}{2} + m + \frac{1}{2}$

$x = n$

and

$y = c - b$

$y = (n - m - \frac{1}{2}) - (m + \frac{1}{2})$

$y = n - m - \frac{1}{2} - m - \frac{1}{2}$

$y = n - 2m - 1$

so $x$ and $y$ are both integers, since integers are closed under subtraction and multiplication.

Therefore, $x$ and $y$ will always be integers, so their product, $xy = a^2$, will always be an integer as well, since integers are closed under multiplication. Thus, any integer can be substituted into $a^2$ and used to construct at least one right triangle, where each right triangle corresponds to a unique $(x,\ y)$ pair.

■

## 5 The Sum Product Table

The Sum Product Table is a program that displays the right triangle for each $(x,\ y)$ pair of integers. The program prompts the user to enter two integer inputs for $x$ and $y$ and generates all triangles in the table up to row $x$ and column $y$. This program further solidifies the understanding that any integer can be represented by a right triangle that illustrates the relationship between the sum and product of its corresponding $(x,\ y)$ pair.

For all $(x,\ y)$,

$a = \sqrt{xy}$

$b = \frac{|x-y|}{2}$

$c = \frac{x+y}{2}$

where $a$ is the height, $b$ is the base, and $c$ is the hypotenuse of the corresponding right triangle.

The program is written in python, and uses the turtle library to display the output. Use the link below to access the code used to generate the table:

The Sum Product Program


Figure 7: A sample output for an input of $x = 8$ and $y = 13$

Not only does this visual representation serve as a geometric proof, but it also gives us insight into the behavior of numbers. The colored boxes indicate where similar triangles (triangles with the same proportions) repeat. For example, all triangles in yellow outlined boxes have a $\theta$ value of 19.47°, which is the angle of the *Kelvin wake pattern* in water. Likewise, triangles in orange boxes have $\theta = 30°$, triangles in pink boxes have $\theta = 36.87°$, and so on. These repeating triangles illustrate how numbers behave in a wave-like pattern. Another interesting observation is that perfect squares cannot create right triangles; when the factors are equal, the difference between them is 0 (having equal geometric and arithmetic means), and therefore $\theta = 0°$. These boxes contain vertical lines instead of right triangles, where $\sqrt{xy} = \frac{x+y}{2}$; these boxes form a diagonal line on the grid, splitting it in half. All triangles in the lower left half of the grid are reflected across the diagonal line in the upper right half of the grid. This is because the integer pair ($x$, $y$) produces the same triangle as the integer pair ($y$, $x$) for all integers $x$ and $y$.

# 6 Complementary Roots

Consider the special case when one perpendicular side is the square root of a quasi-prime number. Note that a quasi-prime number is an integer that can be expressed as exactly two prime numbers, $p_1$ and $p_2$, where $p_1 > 3$ and $p_2 > 3$. The values of $c + a$ and $c$ - $a$ will always be irrational; however, their product will always be a whole number, as shown below. We call this pair of numbers the *complementary roots.* These roots are related to each other through their decimal extensions, always summing to one.

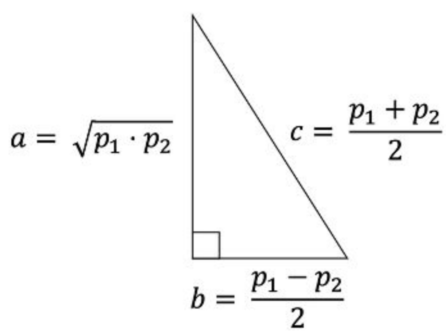


Figure 8: The right triangle configuration for a quasi-prime number, defined by a pair of prime factors

Suppose you are given the quasi-prime number, 30,607. Therefore, 30,607 must have exactly two prime factors, $p_1 = 241$ and $p_2 = 127$. This forms a triangle with sides $a = \sqrt{30,607} = 174.9485638694985...$, $b = \frac{241-127}{2} = 57$, and $c = \frac{241+127}{2} = 184$. The corresponding complementary root values are $c + a =$ 358.9485638694985... and $c$ - $a$ = 9.051436130501488...; the decimal extension of $c + a$ matches that of $\sqrt{30,607}$, and the decimal extensions of $c + a$ and $c$ - $a$ sum precisely to 1: 0.051436130501488... + 0.9485638694985... = 1. The product of the two complementary root values is 3,249. This is a whole number and a perfect square value, with a square root of 57, which is also the length of side $b$.

# 7 The Most Fundamental Geometric Form

As demonstrated by our proof and table, the fact that every number can be represented as at least one right triangle has major implications; this suggests that right triangles are essentially the building blocks of the world around us. In fact, this may be consistent with Plato's *Timaeus*, in which he discusses the formation of our intricately structured universe. He believed that geometric structures birthed the complexity of the universe. *Timaeus* advances this idea by discussing the platonic solids, including the tetrahedron, hexahedron, octahedron, and icosahedron.[3]

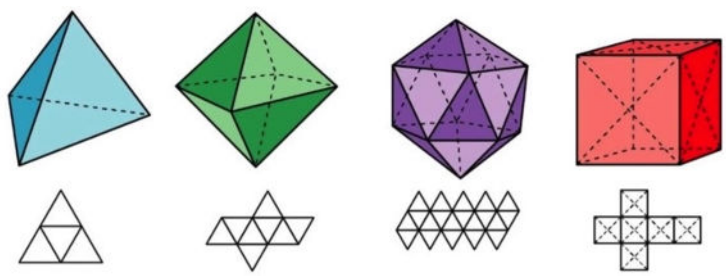

Figure 9: (Left to Right) Tetrahedron, Octahedron, Icosahedron, Hexahedron

Each of these figures can be decomposed into triangles and all triangles (scalene, isosceles, equilateral) can be decomposed into right triangles. Therefore, right triangles are the ultimate building blocks of the world around us.[3] Even right triangles themselves decompose into similar right triangles, which all have the exact proportions of the larger triangle. Thus, right triangles are unique, as they can be infinitely fractalized. Right triangles are the most simple and fundamental geometric form.

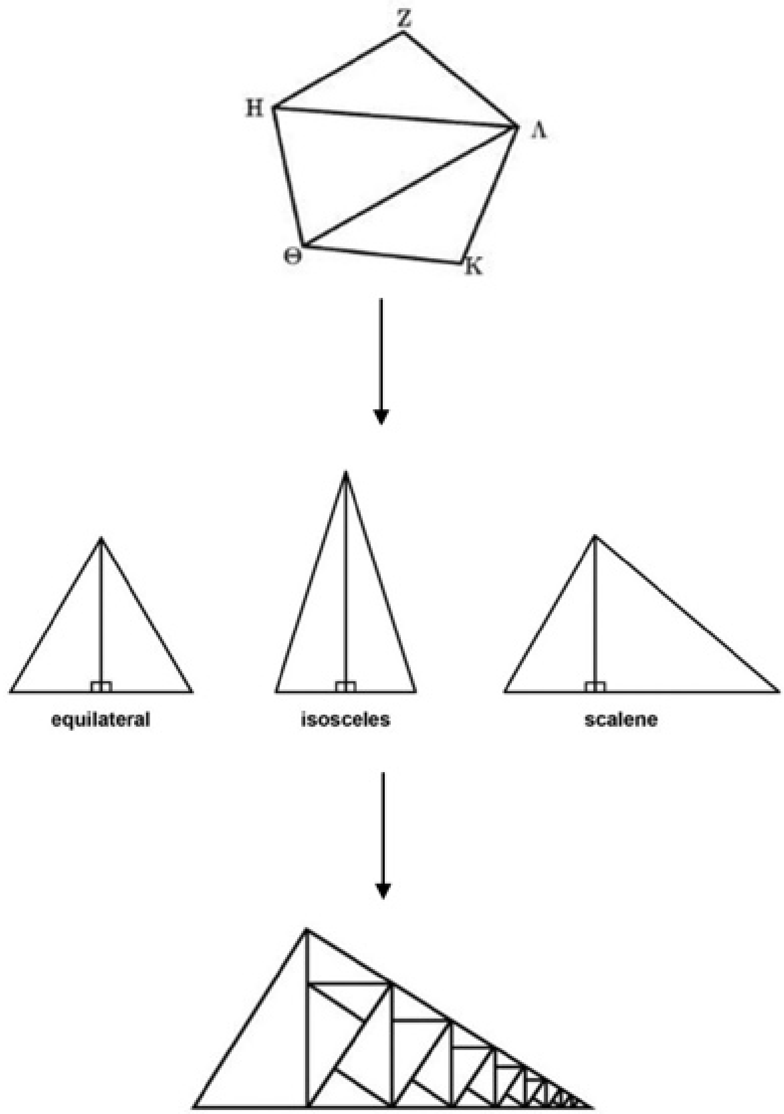


Figure 10: Every object in the physical realm can be deconstructed into a right triangle.[6]

In fact, the most basic unit of a torus is a right triangle. A torus is a donut-like shape that is of great interest in a wide variety of subjects, including geometry and physics. For example, in physics, the torus is considered the best structure for particle acceleration. Additionally, the shape of the universe is hypothesized to be a torus, as this geometric structure is believed to accurately model space-time.[4] Below are the 3 standard torii:

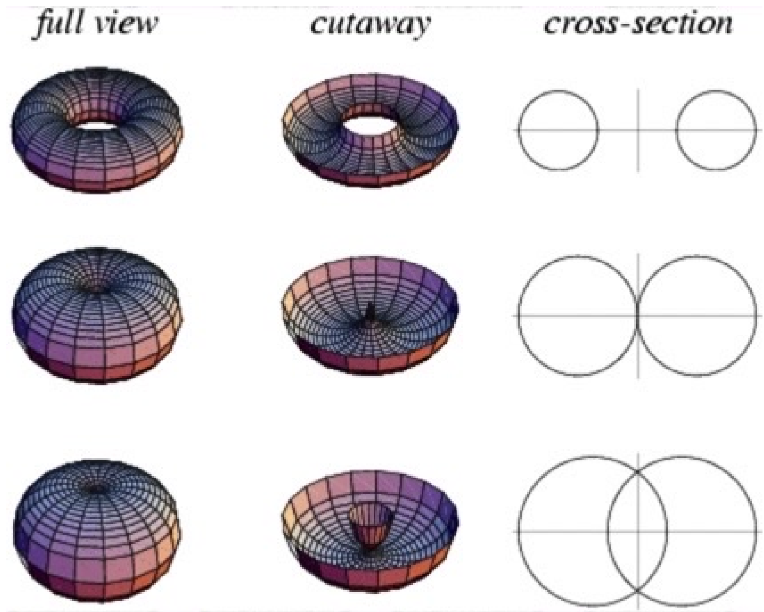


Figure 11: (Top to Bottom) Ring Torus, Horn Torus, Spindle Torus[7]

The torus is defined by the radius of the donut, $R$ and the radius of the tube, $r$. Here is the equation of a torus in 3-dimensional space:

$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$

As shown in the diagram above, the torus can be broken into cross sections. These cross sections can be constructed using the $(x, y)$ pair from its corresponding right triangle.

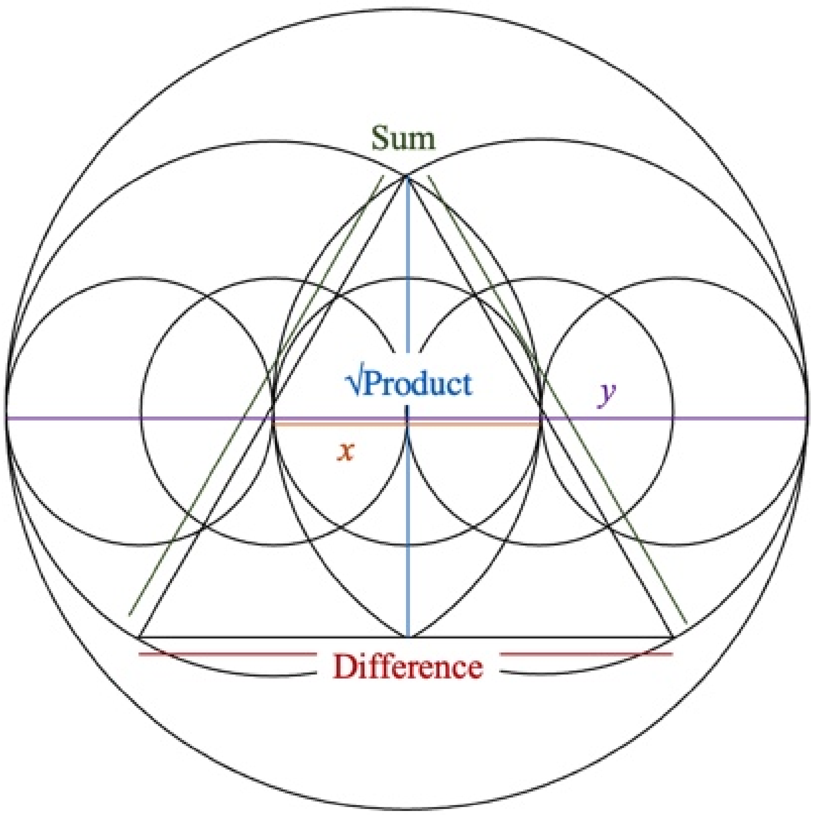


Figure 12: Using the right triangle, we are able to derive $x$ and $y$, which are used to determine $R$ and $r$ of the torus.

# 8 Distinct Sums, Products, and Differences

The Erdős–Szemerédi Theorem mainly concerns the number of distinct sums and distinct products for a given set. More specifically, it focuses on the relationship between the cardinality of the sum grid and the cardinality of the product grid. We can extend our new understanding of the right triangle configuration of sum, product, and difference to cardinalities of sets. As opposed to the right triangle configuration for a pair of numbers, the right triangle configuration for distinct sums, products, and differences of a set is much more fluid, where multiple transpositions are possible. For a pair of numbers, the only right triangle that is possible is the configuration that transposes the sum to the hypotenuse; we will call this the $\Sigma$-*transposition*. However, for cardinalities of sets, other transpositions are possible. The $\Delta$-*transposition* is the right triangle that transposes the difference to the hypotenuse. The final transposition is the $\Pi$-*transposition*, which transposes the product to the hypotenuse.

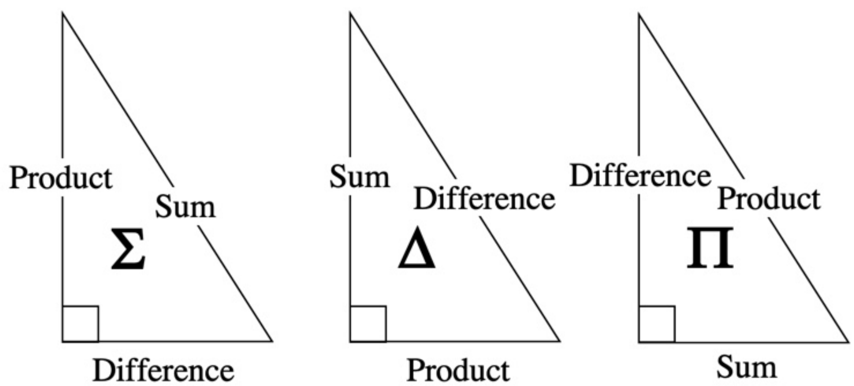


Figure 13: Above are the 3 right triangle configurations: $\Sigma$-*transposition*, $\Delta$-*transposition*, and $\Pi$-*transposition*.

Note that for any given transposition, the height and base are interchangeable; switching the height and base only changes the orientation of a given transposition. Therefore, there are only 3 unique transpositions:

- $\Sigma$-*transposition*, where the sum is used to construct the hypotenuse
- $\Delta$-*transposition*, where the the difference is used to construct the hypotenuse
- $\Pi$-*transposition*, where the product is used to construct the hypotenuse

The fluidity that allows multiple possible right triangle configurations for cardinalities of sets arises from the properties of the original set used to construct the sum, product, and difference grid. This fluidity also allows coefficients to be used to conform to a Pythagorean configuration.

Consider the simple arithmetic progression of integers, {1, 2, 3, 4}. This set has 7 distinct sums, {2, 3, 4, 5, 6, 7, 8}, 9 distinct products, {1, 2, 3, 4, 6, 8, 9, 12, 16}, and 4 distinct differences, {0, 1, 2, 3}. The right triangle configuration of this set is a Δ-*transposition*, where

$$(\sqrt{7})^2 + (\sqrt{9})^2 = (4)^2$$
$$7 + 9 = 16$$

for $a = \sqrt{7}$, $b = \sqrt{9}$, and $c = 4$.

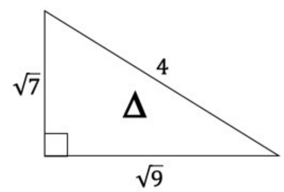


Figure 14: The right triangle configuration for the distinct sums, products, and differences for the set {1, 2, 3, 4}, which is a Δ-*transposition*

**Arithmetic Progression:** {5, 10, 15, 20, 25, 30, 35}. This set has 13 distinct sums, 25 distinct products, and 7 distinct differences. The right triangle configuration of this set is a Δ-*transposition*, where

$$(\sqrt{13})^2 + (\sqrt{25 \cdot 5})^2 = (\sqrt{7 \cdot 9})^2$$
$$13 + 50 = 63$$

for $a = \sqrt{13}$, $b = \sqrt{25 \cdot 5}$, and $c = \sqrt{7 \cdot 9}$.

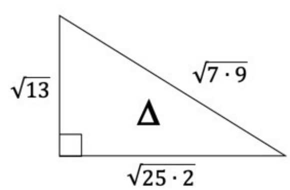


Figure 15: The right triangle configuration for the distinct sums, products, and differences for the set {5, 10, 15, 20, 25, 30, 35}, which is a Δ-*transposition*

**Arithmetic Progression:** {7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77}. This set has 21 distinct sums, 53 distinct products, and 11 distinct differences. The right triangle configuration of this set is a Δ-*transposition*, where

$$(\sqrt{21 \cdot 9})^2 + (\sqrt{53})^2 = (\sqrt{11 \cdot 22})^2$$
$$189 + 53 = 242$$

for $a = \sqrt{21 \cdot 9}$, $b = \sqrt{53}$, and $c = \sqrt{11 \cdot 22}$.

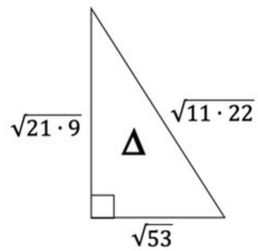


Figure 16: The right triangle configuration for the distinct sums, products, and differences for the set {7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77}, which is a Δ-*transposition*

**Arithmetic Progression:** {11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132, 143}. This set has 25 distinct sums, 72 distinct products, and 13 distinct differences. The right triangle configuration of this set is a Δ-*transposition*, where

$$(\sqrt{25})^2 + (\sqrt{72 \cdot 2})^2 = (13)^2$$
$$25 + 144 = 169$$

for $a = \sqrt{25}$, $b = \sqrt{72 \cdot 2}$, and $c = 13$.

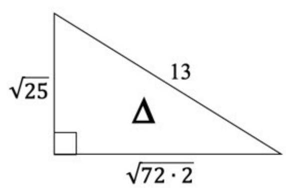


Figure 17: The right triangle configuration for the distinct sums, products, and differences for the set {11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132, 143}, which is a Δ-*transposition*

**Arithmetic Progression:** {13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, 221}. This set has 33 distinct sums, 114 distinct products, and 17 distinct differences. The right triangle configuration of this set is a Δ-*transposition*, where

$$(\sqrt{33 \cdot 2})^2 + (\sqrt{114 \cdot 3})^2 = (\sqrt{17 \cdot 24})^2$$
$$66 + 342 = 408$$

for $a = \sqrt{33 \cdot 2}$, $b = \sqrt{114 \cdot 3}$, and $c = \sqrt{17 \cdot 24}$.

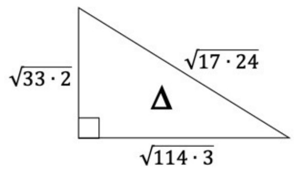


Figure 18: The right triangle configuration for the distinct sums, products, and differences for the set {13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, 221}, which is a Δ-*transposition*

In fact, all arithmetic progressions yield Δ-*transpositions*. Let's see what happens when we try to determine the right triangle configuration of the distinct sums, products, and differences of the geometric progression, {3, 9, 27}. This set has 6 distinct sums, 5 distinct products, and 4 distinct differences. Unlike the previous two examples, the right triangle configuration of this set is a Π-*transposition*, where

$4^2 + (\frac{6}{2})^2 = 5^2$

$16 + 9 = 25$

for $a = 4$, $b = \frac{6}{2}$, and $c = 5$.

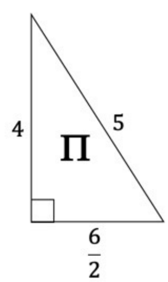


Figure 19: The right triangle configuration for the distinct sums, products, and differences for the set {3, 9, 27}, which is a Π-*transposition*

**Geometric Progression:** {7, 49, 343, 2401, 16807}. This set has 15 distinct sums, 9 distinct products, and 11 distinct differences. The right triangle configuration of this set is a Π-*transposition*, where

$$(\sqrt{11 \cdot 3})^2 + (\sqrt{15 \cdot 2})^2 = (\sqrt{9 \cdot 7})^2$$
$$33 + 30 = 63$$

for $a = \sqrt{11 \cdot 3}$, $b = \sqrt{15 \cdot 2}$, and $c = \sqrt{9 \cdot 7}$.

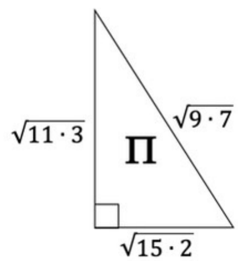


Figure 20: The right triangle configuration for the distinct sums, products, and differences for the set {7, 49, 343, 2401, 16807}, which is a Π-*transposition*

**Geometric Progression:** {11, 121, 1331, 14641, 161051, 1771561, 19487171}. This set has 28 distinct sums, 13 distinct products, and 22 distinct differences. The right triangle configuration of this set is a Π-*transposition*, where

$$(\sqrt{22})^2 + (28)^2 = (\sqrt{13 \cdot 62})^2$$
$$22 + 784 = 806$$

for $a = \sqrt{22}$, $b = 28$, and $c = \sqrt{13 \cdot 62}$.

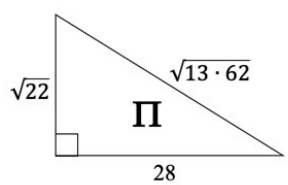


Figure 21: The right triangle configuration for the distinct sums, products, and differences for the set {11, 121, 1331, 14641, 161051, 1771561, 19487171}, which is a Π-*transposition*

**Geometric Progression:** {13, 169, 2197, 28561, 371293, 4826809, 62748517, 815730721, 10604499373, 137858491849, 1792160394037}. This set has 66 distinct sums, 21 distinct products, and 56 distinct differences. The right triangle configuration of this set is a Π-*transposition*, where

$$(\sqrt{56 \cdot 3})^2 + (\sqrt{66 \cdot 7})^2 = (\sqrt{21 \cdot 30})^2$$
$$168 + 462 = 630$$

for $a = \sqrt{56 \cdot 3}$, $b = \sqrt{66 \cdot 7}$, and $c = \sqrt{21 \cdot 30}$.

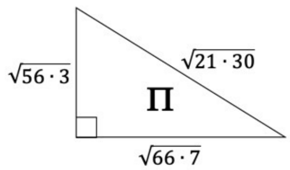


Figure 22: The right triangle configuration for the distinct sums, products, and differences for the set {13, 169, 2197, 28561, 371293, 4826809, 62748517, 815730721, 10604499373, 137858491849, 1792160394037}, which is a Π-*transposition*

**Geometric Progression:** {17, 289, 4913, 83521, 1419857, 24137569, 410338673, 6975757441, 118587876497, 2015993900449, 34271896307633, 582622237229761, 9904578032905937}. This set has 91 distinct sums, 25 distinct products, and 79 distinct differences. The right triangle configuration of this set is a Π-*transposition*, where

$$(\sqrt{79 \cdot 5})^2 + (\sqrt{91 \cdot 5})^2 = (\sqrt{25 \cdot 34})^2$$
$$395 + 455 = 850$$

for $a = \sqrt{79 \cdot 5}$, $b = \sqrt{91 \cdot 5}$, and $c = \sqrt{25 \cdot 34}$.

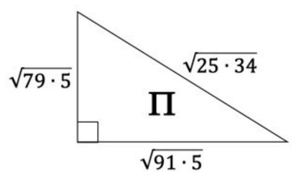


Figure 23: The right triangle configuration for the distinct sums, products, and differences for the set {17, 289, 4913, 83521, 1419857, 24137569, 410338673, 6975757441, 118587876497, 2015993900449, 34271896307633, 582622237229761, 9904578032905937}, which is a Π-*transposition*

We have found that all geometric progressions yield Π-*transpositions*.

## 9 Conclusion

Now that we have defined the direct relationship between the sum of two numbers and their product, we are able to begin addressing many other problems in mathematics. Illustrating this relationship as a right triangle is groundbreaking, and has massive implications. This new visual understanding of numbers gives us a refreshing perspective on how to approach what we don't know about mathematics. We believe that there is still much yet to be uncovered about the universe using this recent discovery.

## 10 References

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