The Fractal Root of Numbers

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Abstract

Formally, the principal roots of numbers, i.e. square or cube roots, etc., are believed to exclusively consist of numerically identical numbers. In this paper, we expand this formal definition by exploring the existence of fractal roots; roots that are identical in their numerical structure but not in their relative magnitude. This new type of root has its own mathematical and geometrical peculiarities with implications intersecting many fields of science.

I. INTRODUCTION

Finding the roots of numbers, such as their square roots, is one of the most fundamental mathematical operations that have been implemented for millennia[1,2].

It is an indispensable tool for solving many important mathematical problems, either algebraic or geometric, such as polynomial equations, standard deviations, the Pythagoras rule for right-angle triangles, etc.

It is also embedded within the formulas of some important physical constants, such as Planck's length[3]: $p_l = \sqrt{\frac{\hbar G}{C^3}}$ (where $\hbar$ is the Planck constant, G is the gravitational constant, and C is the speed of light).

One very familiar root is the square root of number 2, which solves for the hypotenuse of a right-angle triangle whose perpendicular sides measure to unity. Using Pythagoras rule, we can calculate the hypotenuse as: $1^2 + 1^2 = 2 = (\sqrt{2})^2$.

The irrational[4] nature of this number ($\sqrt{2} = 1.41421356\ldots$) is believed to have been confirmed by the Greek mathematician Hippasus of Metapontum (530 - 450 BC)[5]. (This discovery sent shockwaves throughout the Pythagoreans' society who would not accept the existence of such imperfect numbers (i.e. irrational, transcendental, etc.) to the extent they are rumored to have thrown Hippasus into the sea as a punishment for his heretical discovery.)
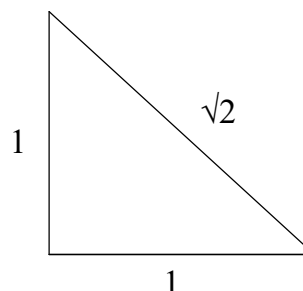

Fig.1: The irrational square root of number 2 defines the hypotenuse of a right-angled triangle having the value of 1 for the other two sides.

It is a consensus between mathematicians that the principal roots of any number must be numerically identical; identical in their numbers sequencing and relative magnitude. They can be negative or positive, but they must be identical in all other aspects.

This restricted the root of any number to one value only, depending on the order of the root.

For example, the square root of number 4 can be number 2 only, plus or minus. Similarly, the cube root of 4 is only $\pm 1.5874\ldots$

But could there be another way to square- or cube-root numbers? In other words, is it possible

to find identical roots for any number, other than its formal roots, such that when multiplied together they will generate the same number again?

At first inspection, this question has an answer of definite *No*. Nevertheless, there could be a possible *yes* answer; however, it requires relaxing the constraints imposed on what we define as *identical* numbers.

II. NUMERICALLY VS. FRACTALLY IDENTICAL

As we explained above, the formal definition for two numbers to be *identical* is that they have the same exact numerical sequencing as well as relative magnitude: their decimal points.

For example, 3.14 is *numerically* identical to 3.14, but it is not considered identical to 0.314 nor to 31.4, etc.

In this paper, we take a bit of a relaxed approach to this strict definition by removing the decimal point constraint. In other words, we consider numbers like 3.14, 0.314 and 31.4 to be *fractally* identical.

This fractal attitude towards numbers originates from the *Wave Theory of Constants*[6] where numbers, specifically mathematical and physical constants, are shown to be fundamental mainly for their unique numerical structure and not for their relative magnitude.

For example, a number like $\pi = 3.14\ldots$ can also be written as 0.314×10 or $31.4/10$, etc. In other words, what really identifies this number is its unique numerical structure: 3, followed by 1, then 4 and so on; it is this specific sequencing that sets this number apart from all other numbers, not its decimal point.

Working with the *fractally-identical* scheme of numbers expands our perception of some mathematical operations and enables the discovery of more possibilities to them.

One such operation is the principal root, which we start investigating its fractal-based possibilities by examining the simplest case: that of the square root.

II. THE FRACTAL ROOT: A NEW KIND OF ROOT

The square root reduces any number *y* to a pair of identical smaller numbers *x* such that $x \times x = x^2 = y$. The result can be an integer or a non-integer (i.e. a float) depending on the original number *y*. We can also reverse the notation to write: $\sqrt[2]{y} = x$.

The number *x* is called the *principal square root* of *y*, which is called the *radicand*. The radical symbol ($\sqrt{\ }$) indicates taking the root of the radicand *y*, and the superscript '2' indicates taking the square root, not the cube or fourth roots, etc. The radical symbol may have its origin from the Arabic letter 'ج', phonetically equivalent to *G* or *J*, which is the first letter in the word '*Jathr*', meaning root, which had been used by Arab mathematicians as a symbol for the root operation[7].

One peculiar square root is that of number 10: $\sqrt[2]{10} = 3.162277\ldots$ This is because the inverse of this number is a fractal of the same number $\frac{1}{\sqrt[2]{10}} = 0.3162277\ldots$

We can approach the above from a different perspective involving the number 1 rather than 10. This is because $3.162277\ldots \times 0.3162277\ldots = 1$.

The formal solution for $\sqrt[2]{1}$ is the two numerically identical principal roots of ± 1. However, we just found two fractally identical numbers, differing only by their decimal points, such that when multiplied together they produce 1 also.

Thus, to find this unique root of number 1, we multiply it by 10, then we find its principal square root *p*. The number 1 will then be equal to $p \times \frac{p}{10}$.

In fact, this unique root of 1 is capable of reproducing the same kind of rooting for any other number.

For example, taking number 5 as the radicand:

$$\sqrt[2]{10 \times 1} \times \sqrt[2]{5} = \sqrt[2]{50} = 7.07106\ldots$$

$$\text{And } 7.07106\ldots \times \frac{7.07106\ldots}{10} = 5.$$

We call this new root the *Square Fractal Root*, and we use the fractal radical sign $\sqrt[fr]{y}$ to indicate its implementation. We also call the value $\sqrt[2]{10 \times y}$ the *greater fractal root* (GFR) of *y*, and the other root (GFR/10) the *lesser fractal root* (LFR).

Both roots can be either positive or negative similar to the principal square root. In fact, these two square roots, the principal and the fractal, are the only square roots a number can have, depending on its order of magnitude. So, for number 5 again, $\sqrt[2]{5} = 2.2360\ldots$ but $\sqrt[2]{0.5} = 0.7071\ldots = \text{LFR}(5)$, $\sqrt[2]{50} = 7.07106\ldots = \text{GFR}(5)$, $\sqrt[2]{500} = 22.3606\ldots$ etc.

Below is a list of the GFR of some constants of interest.

Number	Value	GFR
Pi π	3.140…	5.6049…
Euler *e*	2.718…	5.2137…
Golden Section Φ	1.618…	4.0224…
Number 2	2	4.4721…
Fine Structure α	137	37.013…

Note from the above table how the fractal root of π involves numbers 5 and 6, the numerical references to the pentagon and the hexagon (the two polygons involved in making up the nucleotides, the basic units for the DNA and all life). Interestingly, a pentagon and hexagon separated by a circle (π) that defines their outer and inner boundaries respectively, will have an equal length for their sides as illustrated in figure (2) below.
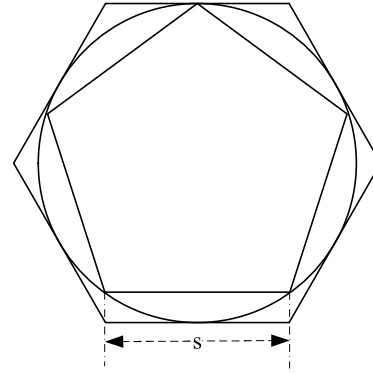

Fig.2: A hexagon and a pentagon, separated by a circle, will have the exact length *s* for their sides.

Thus, the fractal root is a mathematical operation that introduces a different perspective to what we mean by the square root, generating two roots instead of one, with both being identical but for the position of their decimal point.

The square fractal root can be useful in providing different methods for deriving some fundamental physical and mathematical constants. For example, adding 0.3 to the LFR of 1 results in a number almost identical to Planck's length[3] (1.616229×10^{-35} m) minus 1:

$$\frac{\sqrt[fr]{1} + 3}{10} = 0.6162277\ldots$$

This constant is defined as the unit of length that light will travel in one unit of Planck's time (5.39 × 10^{-44} s). It is also the scale at which quantum gravitational effects are believed to become noticeable[8].

Additionally, the fractal root enables us to put forth a geometrical representation of this fundamental constant that is based on a circle of 6-units' diameter and defined by the two values of $\sqrt[fr]{1} - 3$ and $\sqrt[fr]{1} + 3$, as illustrated in figure (3) below.
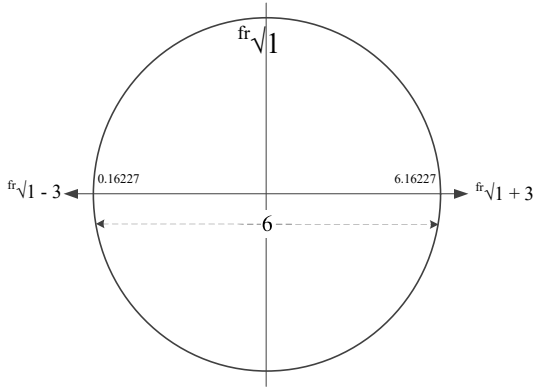
Fig.3: Planck's length, calculated from the fractal root of 1, defined by a circle of 6-units' diameter.

Another usage is to calculate π from the LFR(1) as follows: $\pi = (^{\mathrm{LFR}}\sqrt{1} + 1)^{\frac{1}{0.24}}$. The offset from the real value of π is less than 0.015%.

Additionally, we can calculate e from the fractal roots of 0.718: 2.68... and 0.268... as follows: $e - 2 = 2.68 ... \times 0.268 ....$ Number 2.68 defines one important new constant (*Grant's Constant*) as explained in the Wave Theory of Constants paper[6]. We can also calculate the principal square root of 3 from the LFR(1) as follows: $(LFR(1) + 1)^2 = 1.732 \sim \sqrt{3}$.

Geometrically speaking, the square fractal root maps one shape to another with both shapes having identical areas but with different dimensions. In other words, for some area A, using the principal square root, we assume the area to be that of a square with the sides having equal values (a ratio of 1). By using the fractal root however, we assume the shape to be of a rectangle with its sides fractally identical; having a ratio of 10 instead of 1.
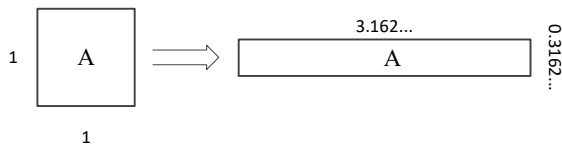

Fig.4: Using the fractal root to transform a square of area A into a rectangle having the same area but with the ratio of the sides equal to 10.

For the case of a circle, the area is calculated as $A = \pi \times r^2$. Using the principal square root, the radius is calculated from the area as $r = \pm\sqrt[2]{\frac{A}{\pi}}$.

The fractal root, on the other hand, generates two fractal values of r. In other words, we are assuming the area A to belong to an ellipse with the major and minor axis (a and b) being the GFR and LFR of A respectively: $a, b = \sqrt[fr]{\frac{A}{\pi}}$.
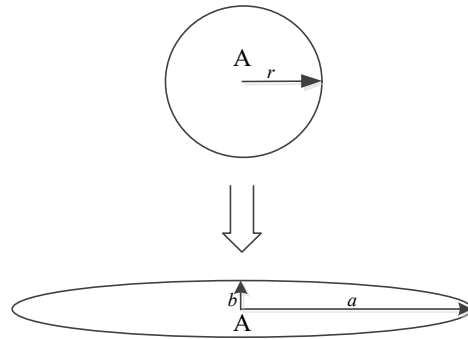

Fig.5: The fractal root transforms a circle of area A into an ellipse of the same area with a/b = 10.

Even though the areas stay unchanged for both shapes, before and after mapping, the perimeters do not.

For example, in the case of the square with unity dimensions, its perimeter is 4×1 = 4. However, the fractally mapped rectangle has dimensions of (3.162... and 0.3162...) producing a perimeter of: (3.162... + 0.3162...) × 2 = 6.9570... > 4.

With the dimensions increasing in value, the perimeters of the square-shaped polygon get bigger in comparison to the rectangular fractal one as shown in figure (6) below.

The perimeters' ratio will equal 1 for the single case of a square's dimension of exactly 3.025 units, as calculated in Appendix A. (For the case of a circle and an ellipse, the magical value is $r = 5.05$ units.)

Interestingly, for dimensions equal to the fundamental constants π and e, the ratios are 1.0191 and 0.948 respectively; both very close to unity.
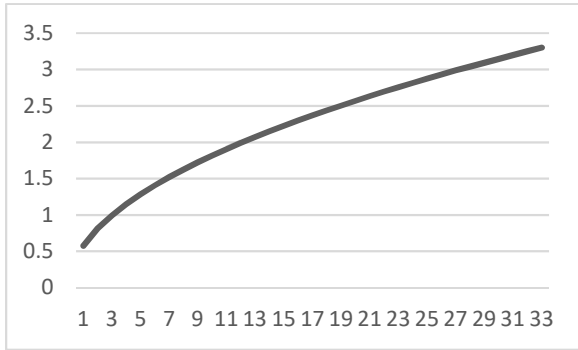
Fig.6: A graph for the ratio between the square and the rectangle's perimeters. A value of 1 occurs at the magical number of 3.025.

There could be some properties for the fractally mapped shapes that make them special to nature in one form or another, similar to those based on the golden ratio Φ[11] for example.

The golden ratio division ensures that the ratio of the whole to the bigger part is equal to that of the bigger part to the smaller one, as shown below.
A rectangle with such ratio for its sides is believed to be harmonious and pleasing to the eye.
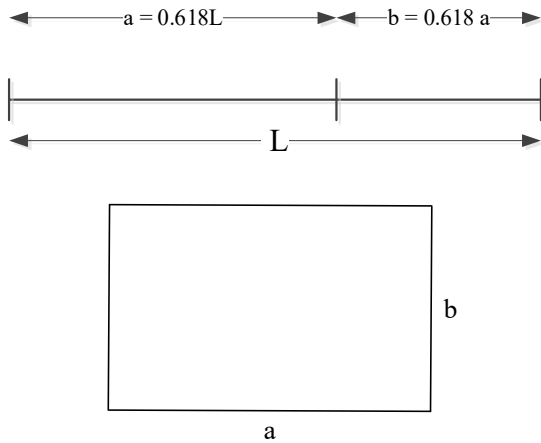

Fig.7: Top: For a given length L, the golden ratio maintains an exact hierarchical proportion for the parts. Bottom: Golden ratio-based rectangle.

For the fractal root, however, the two sides are calculated such that their *multiplication* is equal

to some fixed value while maintaining a ratio of 10 in-between.
Thus, the fractal root is another method that nature can implement to create self-similar fractal copies of itself, numerically as well as geometrically.

III. THE CUBE FRACTAL ROOT

The fractal root is not restricted to square ones; we can also extend it to include cube roots and beyond.
To find the cube fractal root of a number, we start by multiplying it by 100 and then taking its principal cube root.
So, for number 1, the cube root of 100 is equal to 4.6415888…. Multiplying this number with its 1/10 value, twice, results in the number 1 back again:
$$4.64158\ldots \times 0.464158\ldots \times 0.464158\ldots = 1.$$
Thus, similar to the square fractal root, here also we have a couple of roots: the GFR and the LFR (a couple of them).

Below we list the cube-GFRs for the same constants we encountered in the previous section.

Number	Value	GFR
Pi π	3.140…	6.798…
Euler e	2.718…	6.477…
Golden Section Φ	1.618…	5.449…
Number 2	2	5.848…
Fine Structure α	137	23.95…

Notice how the fractal cube root of the fine structure constant is almost equal to 24, a number of particular importance to the Wave Theory of Numbers[9] and Prime Factorization[10].

Geometrically speaking, the fractal cube root transforms a cube into a parallelogram with fractally identical dimensions while keeping the volume the same, as shown in figure (8) below. It also transforms a sphere into an elongated spheroidal shape.
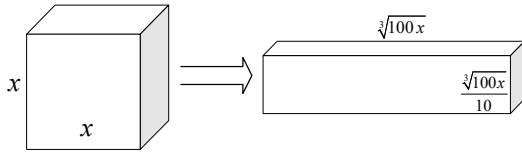
Fig.8: Using the cube fractal root to transform a cube of volume V into a parallelogram having the same volume but with the fractal ratios for the sides.

Similar to the 2-dimensional case where the perimeters were different for the two shapes, in this case, it is the areas that are different with their ratios increasing with dimensionality as shown below.
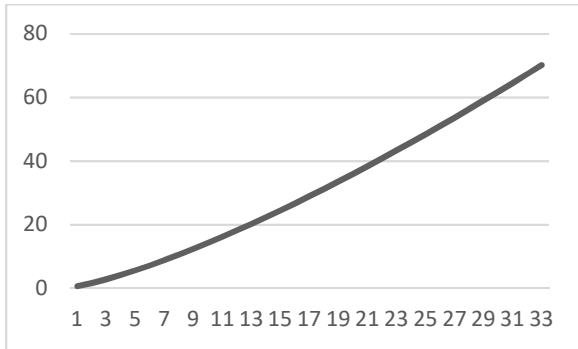

Fig.9: A graph for the ratio between the cube and parallelogram's perimeters. The ratio value of 1 occurs at the magical number of 1.36.

The ratio of the areas is equal to 1 at exactly $x = 1.36089\ldots$ And for the value of 1.37 (a fractal of α) the ratio is 1.009, very close to 1, just like π and e in the square fractal root case.
(The above alludes to a possible relationship between mathematical and physical constants on the one hand, and their corresponding principal-to-fractal perimeters or areas ratios equal to 1 on the other. More testing is definitely needed to confirm or discard this interesting possibility.)

Higher orders of fractal roots, like the fourth or fifth root, etc., can be found in a similar manner. So, for the 4th fractal root case, we multiply the radicand by 1000 and then takes its 4th principal root. The four fractal roots will consist of one GFR and three LFRs.

So, for number 16, we find $\sqrt[4fr]{16} = \sqrt[4fr]{16000} = 12.649\ldots = \text{GFR}(16)$.
Consequently:
$$16 = 12.649\ldots \times 1.2649\ldots \times 1.2649\ldots$$
$$\times 1.2649\ldots$$

Therefore, to calculate the nth fractal root of any number x, we start by multiplying x with 10^{n-1} then we take its principal n^{th} root: $\sqrt[n]{x \times 10^{n-1}}$. The root we get is the GFR of x. The roots always involve a single GFR, along with $n-1$ LFRs. So, 1 LFR for square fractal roots (2-1), 2 LFRs for cube fractal roots, and so on.

Even though the fractal root is mainly derived from applying the principal root operation, still there is a crucial difference between the two.
In the principal root, we can write it in terms of successive roots with smaller orders, something like $\sqrt[4]{16} = \sqrt[2]{\sqrt[2]{16}} = \pm 2$.
However, doing the same for the fractal root produces something entirely different:
$$\sqrt[2fr]{\sqrt[2fr]{16}} = \sqrt[2fr]{12.649\ldots \times 1.2649\ldots}$$
$$= \sqrt[2fr]{12.649\ldots} \times \sqrt[2fr]{1.2649\ldots}$$
$$\sqrt[2fr]{12.649\ldots} = 11.2468\ldots \text{ and } 1.12468\ldots$$
$$\sqrt[2fr]{1.2649\ldots} = 3.556\ldots \text{ and } 0.3556\ldots$$
Now, $0.3556\ldots \times 11.2468\ldots = 4$ and $3.556\ldots \times 1.12468\ldots = 4$ and the total is equal to 16, however, the four roots are not all identical anymore, neither in their number sequencing nor in their decimal point ($i.e.\ 11.2468\ldots$ vs. $3.556\ldots$).
Therefore, not all mathematical operations that work for the principal root work also for the fractal one.

More research is needed to understand the mathematical framework and subtle implications of this new root and where and how it can be implemented.

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IV. CONCLUSION

By expanding the definition of identical numbers to include those having different relative magnitudes, a new type of numbers' root emerges, coined the *fractal root*.

In contrast to the principal root, the fractal root produces a couple of values having the same numeric sequence but with one being ten times the other.

Applications of the fractal root involve providing alternate derivations for fundamental constants as well as mapping geometrical shapes into different ones with fractal ratios for their dimensions.

Analyzing the areas and volumes of the mapped shapes reveals unique dimensional values for the unity-ratio cases.

More work is in progress to reveal other mathematical and geometrical peculiarities of this novel root.

APPENDIX

A- Finding the x dimension that generates a principal-to-fractal perimeter ratio equal to 1:

For a square of sides x and perimeter P_p and fractally mapped rectangle of sides $\sqrt{\frac{x}{10}}$ and $\sqrt{10x}$ and perimeter P_f:

$$P_p = 4 \times x, \quad P_f = \left(\sqrt{\frac{x}{10}} + \sqrt{10x}\right) \times 2$$

For $\frac{P_p}{P_f} = 1 = \frac{4x}{\left(\sqrt{\frac{x}{10}} + \sqrt{10}\right) \times 2} \to 2x = \sqrt{\frac{x}{10}} + \sqrt{10x} \to$

$$2\sqrt{x} = \left(\sqrt{\frac{1}{10}} + \sqrt{10}\right) \to x = 3.025.$$

B- Finding the x dimension that generates a principal-to-fractal area ratio equal to 1:

For a cube of sides x and area A_p and fractally mapped parallelogram of sides $\frac{\sqrt[3]{100x}}{10}$ and $\sqrt[3]{100x}$ and area A_f:

$$A_p = 6 \times x^2, \quad A_f = 2\frac{\sqrt[3]{(100x)^2}}{100} + 4\frac{\sqrt[3]{(100x)^2}}{10}$$

For $\frac{A_p}{A_f} = 1 = \frac{6x^2}{2\frac{\sqrt[3]{(100x)^2}}{100} + 4\frac{\sqrt[3]{(100x)^2}}{10}} \to$

$$3x^2 = \frac{\sqrt[3]{(100x)^2}}{100} + 2\frac{\sqrt[3]{(100x)^2}}{10} \to$$

$$3x^2 = \frac{21}{100}\sqrt[3]{(100x)^2} \to 27x^6 = \left(\frac{21}{100}\right)^3 (100x)^2$$

$$\to 27x^4 = \left(\frac{21^3}{100}\right) \to x = 1.36089\dots$$

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